Fletcher 1979

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1 Mississippi Saxophone

Robert B. Johnston was probably the first to deal with bends and overblows on blues harps from a physical point of view [Jo 87]. In the theoretical part of his 1987 paper he applies formulas from the 1979 paper by *Neville H. Fletcher*, which is about sound generation, in particular on the saxophone [Fle 79]. So the blues harp has its first appearance in the realm of physics as "Mississippi Saxophone"!

Fletcher understands the saxophone in analogy to electrical self-sustained oscillators. In this context he approximately calculates the reed admittance at the transition zone between mouthpiece and bore and relates it to the bore admittance. Johnston treats both reeds in the same channel of a blues harp as a "parallel circuit" of two "saxophone reeds", using Fletcher's approximate admittance formula. The oral cavity substitutes the bore as a resonator (we will discuss this ansatz elsewhere critically in detail).

2 Basic objective of this paper

I have simplified the deduction of the formulas Johnston [Jo 87] uses from Fletcher's treatise [Fle 79], so it might - hopefully - be scanned in as little time and as comprehensible as possible. Obvious imperfections in the original have been corrected. I have tried to improve the notation in places (esp. in chapter 9).

3 Analogy to electric oscillating circuits

Linear acoustics deals with the physics of musical instruments in *analogy to the theory of electricity*. Having understood the theory of alternating current, you will find this efficient and deepening your appreciation of the topic.

Acoustic pressure fluctuations and changes in volume flow are analogous to AC voltage and alternating current. Terms like (complex) impedance and admittance, resistance and reactance are defined accordingly, and many laws of electricity may be used accordingly in acoustics.

> „Analogies are useful when it is desired to compare an unfamiliar system with one that is better known. The relations and actions are more easily visualized, the mathematics more readily applied, and the analytical solutions more readily obtained in the familiar system. Analogies make it possible to extend the line of reasoning into unexplored fields.
>
> All the analysis in musical engineering is concerned with vibrating systems. Although not generally so considered, the electrical circuit is the most common example and the most widely exploited vibrating system. The electrical circuit is a vibrating system in which the kinetic energy, potential energy, and dissipation may be expressed by dynamic equations. This immediately suggests analogies between electrical circuits and other dynamical systems as, for example, mechanical and acoustical vibrating systems." (Harry F. Olson [Ols 67, S. 59f]).

Many musical instruments - especially air-driven instruments - generate sound by means of a self-sustained feedback process. Besides the well-known *Meißner-feedback-circuit* we find electric oscillator circuits, in which an element with *negative differential resistance* (e.g. a *Gunn-diode*) is the cause for a DC source to make up for energy losses (ohmic resistance, emission of electromagnetic waves ...) in the oscillating circuit which defines the frequency. It is common to replace the term negative differential resistance by simply *negative resistance*. The element with negative resistance is called *generator*, the oscillating circuit is called *resonator*.

„Sustained oscillations build up in an LC tank when it is associated with the appropriate negative resistance element. We see, that in both the mathematical and the practical sense, the term *negative resistance* is quite descriptive. Looking at the phenomenon from another angle, we can say that if ordinary or *positive resistance* dissipates power, our negative resistance must provide power. This it does. The negative resistance element derives its power from a d. c. source and yields a portion of this power to the LC tank. We should appreciate that the negative resistance exists as such for the a. c. oscillations. not for the static d. c. voltage and current which define the operating point of the negative resistance element.“ (Irving M. Gottlieb [Got 97, S. 75]).

The saxophone reed functions as a component with negative (differential) resistance: Given a sufficiently high blow pressure, a further pressure increase does not result in a higher but a lower volume flow through the saxophone mouthpiece. This evident property will be quantified in formulas in chapter 4. So, together with a constant blow pressure and the tube as resonator, all three elements of an oscillator that is capable of displaying steady vibration are given: a generator with negative resistance, a resonator and a DC source.

4 The Bernoulli equation

4.1 The stationary Bernoulli equation

A simple and still topical model for the airflow through the reed gap is the Bernoulli equation for stationary, nonviscous and incompressible fluids ([FaBe 12, Abschnitt 5.2]. Neglecting flow velocity inside the oral cavity, and writing v for the velocity inside the reed gap and $p_0 - p$ for the pressure difference between both ends of the (see Fig. 1) one deduces:

$$\frac{\varrho}{2}v^2 = p_0 - p$$

Hence the volume flow $U = |\xi|\, b \cdot v$ through the generator equals:

$$U = |\xi|\, b \cdot \sqrt{2 \cdot \frac{p_0 - p}{\varrho}} \tag{1}$$

The volume flow U is defined to be positive as long as it moves from left to right (see fig. 1). ρ presents the air density (as the Bernoulli equation is valid for imcompressible flows, the density is constant here).

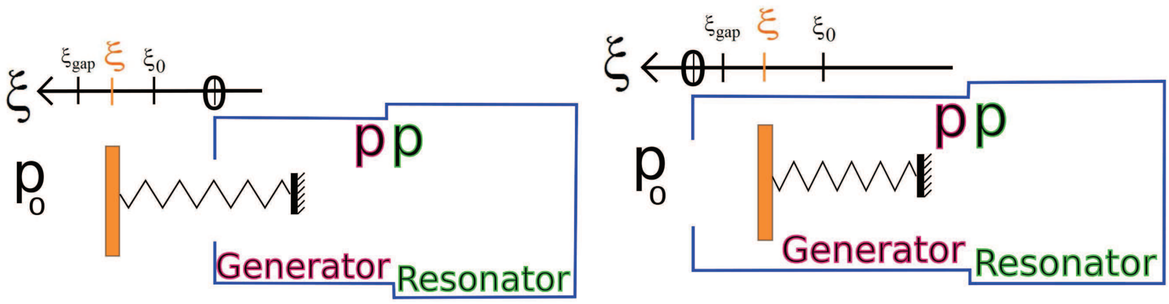

Figure 1: Equivalent oscillator. Left: closing, right: opening valve. The air flows from left to right. Pressures are relative to atmospheric pressure. For $p_0 = 0$ and $p = 0$ the opening equals ξ_{gap}. This also applies more generally to $p_0 = p$. For $p_0 > 0$ and $p = 0$ the opening equals ξ_0. For sinusoidal pressure variations $p = p_1 cos\omega t$ with $p_1 > 0$ the figure shows ξ q ualitatively for $cos\omega t = 1$. Fletcher presupposes $p_0 > p$, which is equivalent to $p - p_0 < 0$. Thus a negative force acts on the spring, which suits the negative displacement $\xi - \xi_{gap}$ from the rest position.

Though Fletcher's formulas are generally conceived for valves including the lips of trumpet players, he speaks mostly of “reeds” (in woodwind instruments or pipe organs). In figure 1 a consistent geometry (vibrating mass on the left, assumed spring on the right) has been chosen for opening and closing reeds. The ξ-axis always leads from right to left (ξ increases when the assumed spring expands). When $\xi = 0$ the generator closes, so that the product $|\xi| \cdot b$ represents the influx area in this one-dimensional model with a suitable parameter b . The opening without pressure from the outside (for saxophone and clarinet this is defined by the player's lip pressure, for the blues harp it is the clearance gap) is called ξ_{gap} (Fletcher: ξ') and is positive for closing reeds, negative for opening reeds.

The oscillating mass is depicted as an harmonic oscillator, with $\xi - \xi_{gap}$ as elongation. If the spring expands, $\xi - \xi_{gap} > 0$ applies for both constellations (opening or closing). Fletcher presupposes $p_0 > p$, which is equivalent to $p - p_0 < 0$. Thus a negative force acts on the spring, which suits the negative displacement $\xi - \xi_{gap}$ from the rest position.

The force exerted on the oscillating mass is calculated as the product $s_r \cdot \Delta p$ of an "effective" surface s_r (r for "reed") and the pressure difference $\Delta p = p_0 - p$. This model presupposes the outer pressure p_0 to be constant. It is alwas bigger than the pressure p at the resonator input. So there is an "oscillating direct voltage" applied across the generator.

At that time measurements had suggested that the Bernoulli equation (1) would have to be generalized for actual implementations. Thus Fletcher adds parameters α und β :

$$U = D\,|\xi|^{\alpha}\,(p_0 - p)^{\beta} \tag{2}$$

In the meantime it has become clear that this generalization is not necessary. So in all formulas of the original study we will substitute $\alpha = 1$ and $\beta = \frac{1}{2}$, e.g.

$$U = D\,|\xi|\,(p_0 - p)^{0,5} \tag{3}$$

using the abbreviation

$$D = b \cdot \sqrt{\frac{\varrho}{2}} \tag{4}$$

4.2 Fletcher's initial term

As a matter of fact there is no stationary flow but an "oscillating DC". Fletcher takes this into account ad hoc by introducing an inertial term. As we have already equated $\beta = \frac{1}{2}$, we can write ξ instead of $|\xi|$:

$$p_0 - p = D^{-2}\xi^{-2}U^2 + M(\xi)\left(\frac{\partial U}{\partial t}\right) \tag{5}$$

Fletcher calculates a masslike load $M(\xi) = \frac{\varrho a}{|\xi| b}$ with a as length of the "reed tip channel" (referred to in table 1 on p. 67 as "effective reed channel length" and given as $2mm$). So the formula for the difference in pressure exerted on the oscillating mass is

$$p_0 - p = D^{-2}\xi^{-2}U^2 + \frac{\varrho a}{|\xi|\, b}\left(\frac{\partial U}{\partial t}\right) \tag{6}$$

4.3 Derivation of the inertial term from the unstationary Bernoulli equation

Taken literally, the derivation in [Fle 79, p. 65] is circular ("we can therefore write, using eq. (9) ..."). Inserting the term for U derived from the stationary Bernoulli equation would lead to $0 = 0 + M(\xi)\left(\frac{\partial U}{\partial t}\right)$ instead of (5).

In order to avoid this flaw it might seem better to start with the *instationary* Bernoulli equation from the very beginning ([Sig, 3.3.6.3.1] or [Kun 90, formula (4.75)]. In our case this results in:

$$\frac{1}{\varrho}p_0 + \frac{0^2}{2} + \int_0^0 \frac{\partial v}{\partial t} ds = \frac{1}{\varrho}p + \frac{v^2}{2} + \int_0^a \frac{\partial v}{\partial t} ds \tag{7}$$

The integrals describe the influence of a local acceleration $\frac{\partial v}{\partial t}$ which takes into account the fluid's inertia for non stationary flows. Presupposing that the temporal velocity change along the short reed tip channel a is constant, (7) implies:

$$p_0 - p = \varrho\frac{v^2}{2} + \frac{\partial v}{\partial t} \cdot a \tag{8}$$

Air flowing with velocity v through the area $|\xi| \cdot b$ represents a volume flow U with $v = \frac{U}{|\xi| \cdot b}$. Together with $D = b \cdot \sqrt{\frac{\varrho}{2}}$ formula (8) implies Fletcher's formula (6).

5 Trigonometric series expansion

Provided the excitation level is not too large one can write elongation, volume flow and pressure as Fourier series with partial frequencies $n\omega$, ω denoting the playing frequency. Fletcher justifies this ansatz in chapter 5 *"Non-linearities"* of his paper, referring to his essay [Fle 78] from the previous year: "If the driving mechanism is sufficiently non-linear ..., all the partials of the sound become locked into exact harmonic relationship...". A more general explanation of this *mode locking* can be found in [Fle 99].

The trigonometric series expansions for elongation, volume flow and pressure are inserted into the generalized Bernoulli equation (6). As Fletcher will ignore higher-order terms later on when deducing the formulas applied by Johnston, we will do so already now for the sake of clarity.

The pressures p_0 und p are measured in relation to the surrounding atmospheric pressure. Pressure p is continuously changing into resonator pressure at the transition between generator and resonator. Within linear acoustics resonator pressure has no constant component. The time-axis is chosen in such a way that the phase shift of pressure p equals zero:

$$p = p_1 cos\omega t \tag{9}$$

Mind the possibly confusing notation for the pressures: Because the static part of pressure p equals zero, index zero may be used for the exterior static pressure p_0 as an abbrevitation. So the cause for the volume flow U ist still the difference in pressure $\Delta p = p_0 - p$ (see 4). Quantities ξ and U show the phases φ and ψ relative to the acoustice pressure fluctuation:

$$\xi = \xi_0 + \xi_1 cos\left(\omega t + \varphi\right) \tag{10}$$

$$U = U_0 + U_1 cos\left(\omega t + \psi\right) \tag{11}$$

As in the long run a comparison of coefficients in terms containing $sinx$ and $cosx$ is pending, we apply addition theorems:

$$\xi = \xi_0 + \xi_1 \left(cos\omega t \cdot cos\varphi - sin\omega t \cdot sin\varphi\right) \tag{12}$$

$$U = U_0 + U_1 \left(cos\omega t \cdot cos\psi - sin\omega t \cdot sin\psi\right) \tag{13}$$

For the mass term in (6) we additionally need:

$$\frac{\partial U}{\partial t} = -U_1 \omega sin\left(\omega t + \psi\right) = -U_1 \omega \left(sin\omega t \cdot cos\psi + cos\omega t \cdot sin\psi\right) \tag{14}$$

6 Oscillator equation

The oscillating mass m_r (r for „reed“) of the generator is modelled as harmonic one-point-oscillator. Influenced by a periodic outer force with angular frequency ω, a steady state solution for the elongation $\xi - \xi_{gap}$ sets in after a short period of transient oscillation. When dealing with the harmonic oscillator in the usual way, the axes for the elongation and for the outer force point in the same direction. Thus a constant outer force extends the elongation. As a constant outer pressure p_0 renders a compression of the spring in fig. 1, $p - p_0$ must be chosen as the difference in pressure for the oscillator equation (contrary to the Bernoulli equation). Inserting (12) and (9) into the equation of motion leads to

$$m_r \left(\frac{d}{dt^2} + \chi_r \omega_r \frac{d}{dt} + \omega_r^2\right) \left(\left(\xi_0 + \xi_1 cos\left(\omega t + \varphi\right)\right) - \xi_{gap}\right) = s_r \left(p_1 \cdot cos\omega t - p_0\right) \tag{15}$$

with χ_r denoting the damping constant, ω_r denoting the eigen frequency of the oscillating mass. The following formula applies for the static part:

$$\xi_0 = \xi_{gap} - \frac{s_r}{m_r \omega_r^2} p_0 \tag{16}$$

For the phase shift φ between elongation and pressure it is well-known that

$$tan\varphi = \frac{\chi_r \omega_r \omega}{\omega^2 - \omega_r^2} \tag{17}$$

with $-\pi \leqq \varphi \leqq 0$, and for the amplitudes:

$$p_1 = \frac{m_r}{s_r} \cdot \xi_1 \cdot \sqrt{\left(\omega_r^2 - \omega^2\right)^2 + \left(\chi \omega_r \omega\right)^2} \equiv K_1 \xi_1 \tag{18}$$

In chapter 8 the sine and cosine values of the phase angle are needed. With (17) and (18) you get

$$cos\varphi = \pm \frac{1}{\sqrt{1 + tan^2\varphi}} = \pm \frac{\omega_r^2 - \omega^2}{\sqrt{\left(\omega^2 - \omega_r^2\right)^2 + \left(\chi_r \omega_r \omega\right)^2}} = \frac{m_r}{K_1 s_r}\left(\omega_r^2 - \omega^2\right) \tag{19}$$

$$sin\varphi = \pm tan\varphi cos\varphi = -\frac{m_r \chi_r \omega_r \omega}{K_1 s_r} \tag{20}$$

When deriving (19) and (20) , the sign depends on the quadrant in which we find the angle φ . Within the range $-\pi < \varphi < 0$ the bracket $\left(\omega_r^2 - \omega^2\right)$ provides for the correct sign for the cosine (for $\omega < \omega_r$ one has $-\frac{\pi}{2} < \varphi < 0$, for $\omega > \omega_r$ one has $-\pi < \varphi < \frac{\pi}{2}$). The sign for $-\pi < \varphi < 0$ is negative throughout the whole interval.

7 Approximation of the Bernoulli equation

Inserting (10), (11) and (14) into the pressure formula (6) results in

$$p_0 - p_1 cos\omega t = D^{-2}\left(\xi_0 + \xi_1 cos\left(\omega t + \varphi\right)\right)^{-2}\left(U_0 + U_1 cos\left(\omega t + \psi\right)\right)^2 + \frac{\varrho a}{\left|\xi_0 + \xi_1 cos\left(\omega t + \psi\right)\right| b}\left(-U_1 \omega sin\left(\omega t + \psi\right)\right)$$

For minor oscillations $|\xi| < \xi_0$ and $U < U_0$ the linear approximation $(1+x)^m = 1 + mx + ...$ may be applied repeatedly in formula (5) :

$$p_0 - p_1 cos\omega t = D^{-2}\left(\xi_0\left(1 + \frac{\xi_1}{\xi_0} cos\left(\omega t + \varphi\right)\right)\right)\left(U_0\left(1 + \frac{U_1}{U_0} cos\left(\omega t + \psi\right)\right)\right)^2 -$$
$$- \frac{\varrho a}{\left|\xi_0 + \xi_1 cos\left(\omega t + \psi\right)\right| b} U_1 \omega sin\left(\omega t + \psi\right)$$

In addition omitting the oscillating part in the denominator of the mass term results in:

$$p_0 - p_1 cos\omega t = D^{-2} \cdot \xi_0^{-2}\left(1 - 2\left(1 + \frac{\xi_1}{\xi_0} cos\left(\omega t + \varphi\right)\right)\right) \cdot U_0^2\left(1 + 2 \cdot \frac{U_1}{U_0} cos\left(\omega t + \psi\right)\right) - \frac{\varrho a}{\left|\xi_0\right| b} U_1 \omega sin\left(\omega t + \psi\right)$$

Because of (6) this formula is valid for the static part:

$$p_0 = D^{-2} \xi_0^{-2} U_0^2$$

Inserting the latter, as well as (12), (13) and (14) leads to

$$p_0 - p_1 cos\omega t = p_0\left(1 - 2\left(1 + \frac{\xi_1}{\xi_0}\left(cos\omega t \cdot cos\varphi - sin\omega t \cdot sin\varphi\right)\right)\right) \cdot \left(1 + 2 \cdot \frac{U_1}{U_0}\left(cos\omega t \cdot cos\psi - sin\omega t \cdot sin\psi\right)\right) -$$
$$- \frac{\varrho a}{\left|\xi_0\right| b}\left(U_1 \omega\left(sin\omega t \cdot cos\psi + cos\omega t \cdot sin\psi\right)\right) \tag{21}$$

We assume now an expansion of all products and collect terms with *cosωt*:

$$-p_1 cos\omega t = 2 \cdot p_0\left(\frac{U_1}{U_0} cos\omega t \cdot cos\psi - \frac{\xi_1}{\xi_0} cos\omega t \cdot cos\varphi\right) - \frac{\varrho a}{\left|\xi_0\right| b} U_1 \omega cos\omega t \cdot sin\psi$$

Inserting $\xi_1 = \frac{p_1}{K_1}$ with K_1 from (18) and comparison of coefficients leads to::

$$-p_1 = 2p_0 \left(\frac{U_1}{U_0} cos\psi - \frac{p_1}{\xi_0 K_1} cos\varphi \right) - \frac{\varrho a}{|\xi_0|\, b} U_1 \omega sin\psi$$

If rearranged, this is Fletcher's formula (20) (with $\alpha = 1$ and $\beta = \frac{1}{2}$, thus $\frac{\alpha}{\beta} = 2$) and $\zeta = 0$):

$$p_1 \left(\frac{2p_0}{\xi_0 K_1} cos\varphi - 1 \right) = U_1 \frac{2p_0}{U_0} cos\psi - \frac{\varrho a}{|\xi_0|\, b} U_1 \omega sin\psi \tag{22}$$

Now we assume again (21) being expanded, collect terms with $sin\omega t$, replace $\xi_1 = \frac{p_1}{K_1}$, compare the coefficients and rearrange:

$$0 = 2p_0 \left(-\frac{U_1}{U_0} sin\omega t \cdot sin\psi + \frac{\xi_1}{\xi_0} sin\omega t \cdot sin\varphi \right) - \frac{\varrho a}{|\xi_0|\, b} U_1 \omega sin\omega t \cdot cos\psi$$

$$0 = 2p_0 \left(\frac{U_1}{U_0} sin\psi - \frac{p_1}{\xi_0 K_1} sin\varphi \right) + \frac{\varrho a}{|\xi_0|\, b} U_1 \omega cos\psi$$

$$p_1 \frac{2p_0}{\xi_0 K_1} sin\varphi = U_1 \frac{2p_0}{U_0} sin\psi + \frac{\varrho a}{|\xi_0|\, b} U_1 \omega cos\psi \tag{23}$$

8 The tangens formula

Inserting (19) and (20) into (22) and (23) results in the following system of equations for phase shifts $-\pi < \varphi < 0$:

$$p_1 \left(\frac{2p_0}{\xi_0 K_1} \frac{m_r}{K_1 s_r} \left(\omega_r^2 - \omega^2\right) - 1 \right) = U_1 \left(\frac{2p_0}{U_0} cos\psi - \frac{\varrho a}{|\xi_0|\, b} \omega sin\psi \right) \tag{24}$$

$$-p_1 \frac{2p_0}{\xi_0 K_1} \frac{m_r \chi_r \omega_r \omega}{K_1 s_r} = U_1 \left(\frac{2p_0}{U_0} sin\psi + \frac{\varrho a}{|\xi_0|\, b} \omega cos\psi \right) \tag{25}$$

Solving (24) and (25) for $\pm \frac{p_1}{U_1}$ and subsequent addition leads to

$$0 = \frac{\frac{2p_0}{U_0} cos\psi - \frac{\varrho a}{|\xi_0| b} \omega sin\psi}{\frac{2p_0}{\xi_0 K_1} \frac{m_r}{K_1 s_r} \left(\omega_r^2 - \omega^2\right) - 1} + \frac{\frac{2p_0}{U_0} sin\psi + \frac{\varrho a}{|\xi_0| b} \omega cos\psi}{\frac{2p_0}{\xi_0 K_1} \frac{m_r \chi_r \omega_r \omega}{K_1 s_r}}$$

- -

$$-\frac{\frac{2p_0}{U_0} sin\psi + \frac{\varrho a}{|\xi_0| b} \omega cos\psi}{A} = \frac{\frac{2p_0}{U_0} cos\psi - \frac{\varrho a}{|\xi_0| b} \omega sin\psi}{B - 1}$$

$$(B-1) \frac{2p_0}{U_0} sin\psi + (B-1) \frac{\varrho a}{|\xi_0|\, b} \omega cos\psi = -A \frac{2p_0}{U_0} cos\psi + A \frac{\varrho a}{|\xi_0|\, b} \omega sin\psi$$

$$sin\psi \left((B-1) \frac{2p_0}{U_0} - A \frac{\varrho a}{|\xi_0|\, b} \omega \right) = cos\psi \left(-\frac{2p_0}{U_0} A - (B-1) \frac{\varrho a}{|\xi_0|\, b} \omega \right)$$

$$tan\Psi = \frac{-\frac{2p_0}{U_0} A - \omega \frac{\varrho a}{|\xi_0| b} (B-1)}{-\frac{2p_0}{U_0} (1-B) - \omega \frac{\varrho a}{|\xi_0| b} A}$$

$$tan\Psi = \frac{\frac{2p_0}{U_0} A + \omega \frac{\varrho a}{|\xi_0| b} (B-1)}{\frac{2p_0}{U_0} (1-B) + \omega \frac{\varrho a}{|\xi_0| b} A} \tag{26}$$

This is the tangens formula (21) from Fletcher 79 (cf. err. in [Fle 82]) with

$$A = \frac{2m_r p_0}{\xi_0 K_1^2 s_r} \chi \omega_r \omega \tag{27}$$

$$B = \frac{2m_r p_0}{\xi_0 K_1^2 s_r} \left(\omega_r^2 - \omega^2\right) \tag{28}$$

9 The admittance of the generator

Looking from the resonator towards the generator, you get as complex admittance $\hat{Y}_r$ of the generator:

$$\hat{Y}_r = -\left(\frac{\partial U}{\partial p}\right)_{p_0} = -\frac{\hat{U}}{\hat{p}} \tag{29}$$

Here $\hat{U}$ and $\hat{p}$ are the sinusoidal components (11) and (9) with frequency ω of volume flow and pressure in complex notation. For $\hat{Y}_r$ Fletcher does not use the usual notation $\left|\hat{Y}_r\right| \cdot e^{i(\omega t+\psi)}$ with $Y_r = \left|\hat{Y}_r\right| \geq 0$ and $0 \leq \psi < 2\pi$. Instead he covers the entire complex plane by limiting the phase to $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$ and at the same time allowing the factor in front of the exponential function to be negative. For this possibly negative factor he uses $[Y_r]$ and calls it „magnitude“. Johnston follows Fletcher, writing however Y_r instead of $[Y_r]$ and calling it „admittance“. From (24) the result with these definitions is

$$[Y_r] = \frac{1-B}{\frac{2p_0}{U_0}cos\psi - \frac{\varrho a}{|\xi_0|b}\omega sin\psi}$$

What we finally need in chapter 10 are the real and the imaginary part of the complex admittance, which are calculated in analogy to the usual formulas:

$$Re\hat{Y}_r = [Y_r] \cdot cos\psi \tag{30}$$

$$Im\hat{Y}_r = [Y_r] \cdot sin\psi \tag{31}$$

Taking the fourth quadrant as example, we can see that Fletcher in fact achieves the real and the imaginary part with his conventions: Here we have $-\frac{\pi}{2} \leq \psi \leq 0$ (thus $cos\psi > 0$ and $sin\Psi < 0$), and $[Y_r] < 0$. So all in all there is a negative real part and a positive imaginary part.

In chapter 10 formulas like $Im\left(\hat{Y}_r + \hat{Y}_p\right) = 0$ are used (p for „pipe“). Should the admittance $\hat{Y}_p$ of the resonator be calculated with the usual definition of modulus and phase, in contrast to Fletcher's convention, both imaginary parts must be calculated independently and then added: $Im\hat{Y}_r + Im\hat{Y}_p = 0$.

10 When does the instrument sound?

At the borderline between generator and resonator (i.e. in the saxophone: inside the mouthpiece) an acoustic pressure $\hat{p} \sim e^{i\omega t}$ is built up, which produces an acoustic volume flow $\hat{U}$ in the resonator through the generator. Then $\hat{U} = \hat{p} \cdot \hat{Y}_r$ flows from the generator, while "negative energy" $p^2 Re\left(-Y_r\right)$ occurs in the generator (with p^2 as the square of the modulus). This energy may now be dissipated in the resonator as $p^2 ReY_p$. In this context "negative energy" is to be understood as used in AC physics: With a corresponding phase shift of current I and voltage U the formal product $U \cdot I \cdot \Delta t$ may be negative during a time periode Δt (see e.g. [Got 97, S. 4ff]).

Thus a self-sustaining acoustic feedback is ceated, while energy arising from the player's expiration is transferred to the resonator by means of the generator. Losses in the resonator (friction, emission of acoustic energy, ...) are being replaced with this additional energy. At the same time the acoustic pressure created at the borderline between generator and resonator sustains the oscillation of the generator and replaces the damping losses there, so that the energy needed for this eventually comes, just the same, from the players's expiration.

De facto, steady oscillations are setting in because of additional non-linear influences which are not depictied in the simple model. Here inpult and output of energy are balanced out exactly. The real parts of the admittances for generator and resonator must be exactly opposite:

$$Re\left(\hat{Y}_r + \hat{Y}_p\right) = 0 \tag{32}$$

Then also the imaginary parts of the acoustic impedances must be exactly opposite, so that the same pressure multiplied by the respective admittances of generator and resonator results in the same volume flow:

$$Im\left(\hat{Y}_r + \hat{Y}_p\right) = 0 \tag{33}$$

Fletcher points out that the second condition (33) is crucial for the playing frequency of the instrument, while the first condition (32) may be fulfilled by means of the force of blowing.

In a later publication [Fle 82] Fletcher is able to verify his theoretical predictions.

11 Outlook

A much simpler formula for the admittance in comparison to (30) and (31) can be found in [Ro 07, (15.128)] :

$$Y\left(\omega\right) = Y\left(0\right)\frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2+\frac{i\omega}{\omega_0 Q}} \tag{34}$$

$Y\left(0\right)$ is the quasistatic flow admittance (or conductance), which is calculated from the stationary Bernoulli equation. The fracture describes the dynamic resonant response of the reed. Further details will be available on www.bluesharpscience.de elsewhere.

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