Linear stability analysis applied to a blues harmonica

Alfred Förtsch*

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1 Introduction

Robert B. Johnston is probably the first to make bending and overblow on the blues harp the subject of physical investigations [Jo 87]. In the theoretical part of his 1987 work, he uses formulas from a paper published in 1979 by Neville H. Fletcher, which deals in particular with sound generation of saxophones [Fle 79]. The blues harp has its first appearance in physics as "Mississippi Saxophone"!

In this paper we will derive and discuss an explicit formula for the admittance of the reeds system in the channel of a blues harp. Included will be a flow component caused by the movement of the reed surfaces. Fletcher's "inertial term" will be neglected. Looking for possible playing frequencies within the framework of linear stability analysis, real and imaginary parts of the admittances of the reeds system and of the vocal tract as resonator have to match. To my knowledge, there are neither measurements nor mathematical models for the admittance of the vocal tract. Therefore, we will use a single-mode approximation for the resonator as a toy model.

Several examples will discuss the existence of drawbends, blowbends and overblows on channel #4 of a C-harp, which represents a typical reed system on a lower channel #1 to #6 of a blues harmonica. A final discussion will compare theoretical results to playing experience. Positive results, but also limits of linear stability analysis applied to tone generation on the blues harp are addressed. In particular, the onset of a self-excited feedback process probably requires analysis in time domain.

2 Derivation of an admittance formula

Blues harps

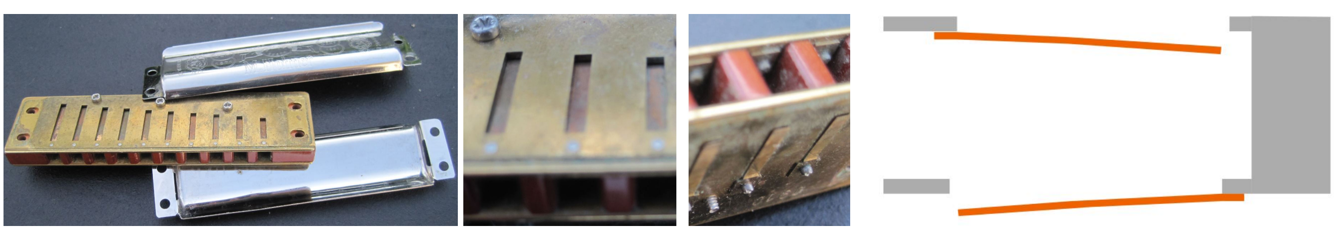

Figure 1: From left to right: Blues harmonica with comb, reedplates and coverplates (unscrewed). Blow reeds. Draw reeds. Schematic sectional drawing through a channel with blow reed (top) and draw reed (bottom), with the reed gap drawn exaggeratedly enlarged in each case.

A blues harmonica (blues harp)[1] is a diatonic harmonica with ten channels [BAB98]. In each channel there is a blow reed and a draw reed. The reeds are riveted to the reedplates and can oscillate freely through openings in the reedplates (free reeds). The blow reeds are located on the underside of the upper reedplate, i.e. inside

*bluesharp-foe@t-online.de

[1] In the following we assume a standard blues harp in Richter tuning

the channels. The draw reeds are located on the underside of the lower reedplate, i.e. outside the channels. For the instrument to be playable, the reed tips must protrude slightly from the reedplate (clearance gap). Blowing into the channel, the blow reed initially behaves (as long as it does not enter the opening in the reedplate) as a closing reed and the draw reed behaves as an opening reed[2]. When one starts drawing, the blow reed behaves as an opening and the draw reed as a closing reed.

Blues harp model

We will be guided by the chapter about "Self-sustained oscillations of a clarinet" in [RH 04, 5.4] and use similar terminology. A consequent distinction between quantities in time (small symboles: $p\,(t)$...) and in frequency domain (large symbols: $P\,(\omega)$...) such as in [ChK 08, ChK 16, Sil 09] will not be used, upper and lower case letters have no deeper meaning in the following.

Our derivation will be based on Fig. 2 which shows a model of a reed channel with two reeds. Free reeds inside a blues harp oscillate nearly sinusoidally [Mi 99] and can therefore be treated as abstract 1-point oscillators as depicted. We will use the index j to distinguish between blow reed ($j = 1$) und draw reed ($j = 2$). The tip of reed j is a distance h_j away from the reed plate (indices are omitted in Fig. 2). This distances h_j are at the same time the elongations of the equivalent one-point oscillators modeling the oscillating reeds. If no pressure is applied to the harmonica, h_j equals the respective clearance gap $h_{j,gap}$. The reeds oscillate around mean elongations h_{jo}. Note that the blow reed (upper reedplate) is a closing reed and the draw reed (lower reedplate) is an opening reed for blow notes. For draw notes the draw reed is closing and the blow reed is opening.

The oscillating air flow is modeled in one dimension. Somewhere inside the reed channel, near the reeds, a boundary is assumed which separates the reeds system from the "resonator" (dashed line in Fig 2). The resonator volume includes part of the reed channel, the player's vocal tract, as well as parts of the trachea (windpipe) below the larynx (voice box).

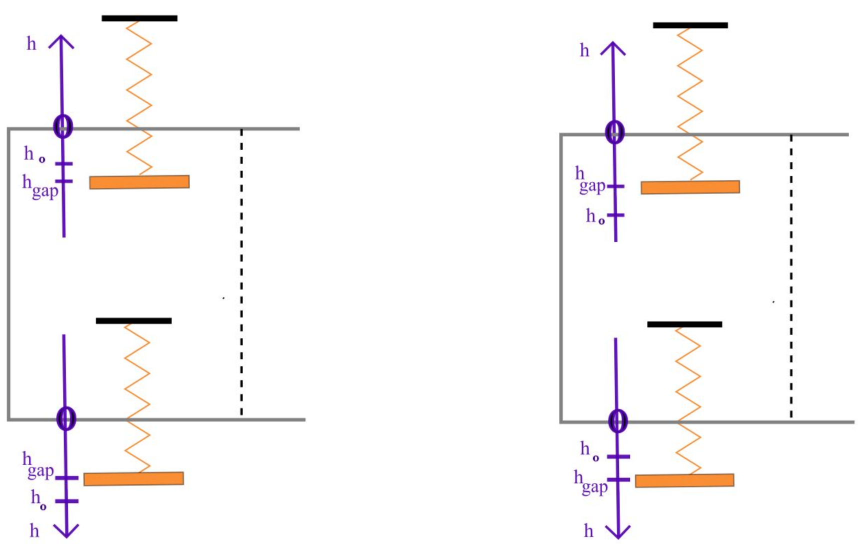

Figure 2: Blues harp model. Left: Blow notes. Right: draw notes. Upper oscillator: blow reed. Lower oscillator: draw reed. Dashed line: boundary towards the resonator.

Pressure

Between resonator and reeds system, pressure $p\,(t)$ relative to atmospheric pressure is assumed to be homogeneous. It is positive for blow notes and negative for draw notes. Compared to the pressure fluctuations inside the channel, the emitted sound pressure outside the reedplates can be neglected [Mi 99]. So each of the reeds "feels" a pressure difference equal to $p\,(t)$.

[2]Nomenclature following Johnston [Jo 87]

Reed motion

Reed parameters are m_j = effective mass of the corresponding oscillator, γ_j =elasticity and K_j = stiffness (so that $\omega_{j0}^2 = K_j/m_j$ with eigen frequency ω_{j0} of reed j). The two reeds in a given channel are rather similar in shape, so it is reasonable to assume common effective surfaces $S_j = S$ as well as common width W (see the Appendix for the discussion of effective quantities).

Pressure $p(t)$ inside the channel will act on both reed surfaces with common force $S \cdot p$. So the equations of motion for the reeds ($j = 1, 2$) are:

$$m_j \frac{d^2 h_j}{dt^2} + \gamma \frac{dh_j}{dt} + K_j \left(h_j - h_{gap}\right) = S \cdot p \tag{1}$$

The directions of the axes in Fig. 2 and the positive sign on the right hand side of equation (1) match in the sense that constant pressures p_0 result in elongations h_{j0} as drawn.

[RH 04] considers harmonic pressure fluctuations $p = p_0 + \hat{p} \cdot \exp i\omega t$, allowing complex “angular frequency” ω. Measured pressures are real parts Rep of complex pressures p. The origin $t = 0$ of the time scale is chosen so that pressure amplitude $\hat{p}$ is real (pressure has phase angle zero). The constant pressure p_0 will be positive for blow notes and negative for draw notes, and it is assumed that $|p_0| > \hat{p}$, i.e. $Rep\,(t) > 0$ for blow notes and $Rep\,(t) < 0$ for draw notes throughout. The equation of motion (1) is fullfilled for pressure fluctuations $p = p_0 + \hat{p} \cdot \exp i\omega t$, if:

$$h_j = h_{j0} + \hat{h_j} \cdot \exp i\omega t \tag{2}$$

$$h_{j0} = h_{j,gap} + \frac{Sp_0}{K_j} \tag{3}$$

$$\left(-\omega^2 m_j + i\omega\gamma_j + K_j\right) \hat{h_j} = S\hat{p} \tag{4}$$

For $Im\omega = 0$ (real-valued ω) real parts of solutions of the form (2) are phase-shifted harmonic oscillations with angular frequency ω and mean opening h_{j0}. Phase shift is encoded in the complex amplitude $\hat{h_j}$ as an angle in the complex plane.

For $Im\omega < 0$ this amplitude increases exponentially without limit, for $Im\omega > 0$ it decreases exponentially towards zero.

Real parts

For the sake of completeness, we add explicit formulas for the measurable real values of openings for sinusoidal oscillations (real-valued ω).

The solution of (4) is:

$$\hat{h_j} = \frac{S\hat{p}}{-\omega^2 m_j + i\omega\gamma_j + K_j}$$

With $\omega_{j0}^2 = K_j/m_j$ (with effective mass m_j and reed eigen frequency ω_{j0}), and damping constant q_j which is defined by $\gamma_j = m_j \omega_{j0} q_j$, it follows:

$$\hat{h_j} = \frac{S\hat{p}}{m_j} \cdot \frac{\left(\omega_{j0}^2 - \omega^2\right) - i\omega_{j0}\omega q_j}{\left(\omega_{j0}^2 - \omega^2\right)^2 + \left(\omega_{j0}\omega q_j\right)^2}$$

With $a = \omega_{j0}^2 - \omega^2$, $b = \omega_{0j}\omega_j q_j$, the real part $h_{r,osc}$ of the oscillating reed opening component $\hat{h_j} \cdot \exp i\omega t$ is thus proportional to $a \cdot \cos \omega t + b \cdot \sin \omega$:

$$h_{r,osc} \propto a \cdot \cos \omega t + b \cdot \sin \omega t = \frac{a}{\cos\left(\arctan \frac{b}{a}\right)} \cdot \cos\left(\omega t - \arctan \frac{b}{a}\right) \tag{5}$$

For $\omega < \omega_0$, reed opening oscillations $\propto \cos(\omega t + \varphi)$ follow pressure fluctuations in time with phase shifts $\varphi \in \left]-\pi/2, 0\right[$. Because of the negative sign of a in (5), $\varphi \in \left]-\pi, -\pi/2\right[$ for $\omega > \omega_0$.

Air velocity

In what follows, we will take a rather crude one-dimensional lumped model approach to a complicated three-dimensional process. It will be assumed that the air flow enters the gap between reed and reed plate with negligible velocity measured at some distance ahead of the gap, but leaves it with high speed. Along its way through the gap constant air density ϱ of the air stream will be assumend and all together, we will neglect inertia effects so that Bernoulli's stationary equation can be applied. Schlieren visualizations show that the air flow is turbulent outside the reed plate, so there is no pressure recovery (in the sense of Bernoulli's equation). This reasoning is essentially based on the fact that the slots between the reed and the reedplate are narrow.

For a blow note, air will flow from the player's vocal tract into the reed channel and from the reed channel through the two slits between the reeds and the reed plates into the surroundings. For draw notes the direction of the air flow is reversed. The velocity at the boundary between reed channel and vocal tract will be denoted by v, the associated volume flow by Q. At slit number j there is velocity v_j and volume flow Q_j.

Because only linear harmonic distortions will be discussed which are assumed to be very small compared to the respective mean values, velocities and volume flows will have one unique direction for blow resp. draw notes. As we are going to calculate *input* admittances for single reeds and for the reeds system inside a harp channel we will chose the *velocity and volume flow axes to be directed from the vocal tract to the inside of the channel.* Therefore, velocities and volume flows will be positive for blow notes and negative for draw notes (as long as they are described by real numbers).

Applying Bernoulli's stationary equation to the air flow through the slits between reed number j and reedplate, the symbol Δp will denote real-valued pressure fluctuations for a moment (think e.g. of $\Delta p \propto \sin \omega t$). For small fluctuations Δp the magnitude of the pressure inside the channel is $p_0 + \Delta p$ with $p_0 > 0$ for blow notes and $|p_0| - \Delta p$ with $p_0 < 0$ for draw notes (see Fig. 3 for an example, magnitude equal to distance from the horizontal axis).

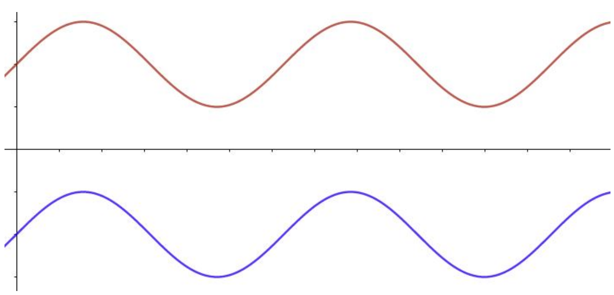

Figure 3: Shifted sine functions $y = 2+\sin x$ and $y = -2+\sin x$ with distances $2+\sin x$ resp. $-(-2+\sin x) = 2-\sin x$ from the x-axis

So, air velocity v_j at reed j equals (with a plus sign for blow notes and a minus sign for draw notes):

$$v_j = \pm\sqrt{\frac{2\left(|p_0| \pm \Delta p\right)}{\rho}} \tag{6}$$

The function $f(x) = \sqrt{a \pm x}$ can be linearly approximated around $x = 0$ by:

$$\sqrt{a \pm x} = f(0) \pm f'(x)\,|_{x=0} \cdot x \pm ... = \sqrt{a} \pm \frac{1}{2\sqrt{a}} \cdot x \pm ... \tag{7}$$

Applied to fomula (6) for the velocites inside the slits one gets:

$$v_j = \pm\sqrt{\frac{2}{\varrho}} \cdot \left(\sqrt{|p_0|} \pm \frac{1}{2\sqrt{|p_0|}} \cdot \Delta p\right) \tag{8}$$

This is a good approximation for large enough values of $|p_0|$ (we will use e.g. $p_0 = \pm 700Pa$ below). It is mathematically also well defined for harmonic perturbations with complex values $\hat{p} \exp i\omega t$:

$$v_j = \pm\sqrt{\frac{2}{\varrho}} \cdot \left(\sqrt{|p_0|} \pm \frac{1}{2\sqrt{|p_0|}} \cdot \hat{p} \exp i\omega t\right) \tag{9}$$

Air velocity v_j can therefore be written as $v_j = v_{j0} + \hat{v}_j \cdot \exp i\omega t$ with mean velocity v_{j0} and amplitude $\hat{v}_j$:

$$v_{j0} = \pm\sqrt{\frac{2\,|p_0|}{\varrho}} \tag{10}$$

$$\hat{v}_j = \sqrt{\frac{2}{\varrho}} \cdot \frac{1}{2\sqrt{|p_0|}} \cdot \hat{p} \tag{11}$$

No instationary Bernoulli equation

[Fle 79, Jo 87] use admittance formulas based on an instationary Bernoulli equation. The extra "inertial term" can surely be neglected for channel #4 of a C-harp [Fö-J] and is no longer used for woodwind instruments either [FR 98, ChK 08, ChK 16].

One should, however, note that instationary Bernoulli equations are essential in nonlinear modelling of the accordion [RCM 05]. The inertia of the flowing air might also be responsible for the generation of normal notes on the blues harp.

Volume flow

The volume flow Q inside the reed channel can be assumed to be homogenous because the reed channel is short compared to musical wavelengths. This volume flow is thought to split into two flows Q_j in the neighbourhood of the two reeds. Each Q_j is the sum of a flow through the respective reed gap plus a "pumped flow" caused by the moving reed surface, which acts similar to the membrane of a loudspeaker.

Thus the volume flow at the boundary between reed channel and vocal tract equals the sum of the volume flows Q_i through both reed gaps. For blow notes, Q_i will be the volume flow leaving the gap between reed and reedplate at the outside. For draw notes, Q_i is the flow entering the reed channel at the inside. In each case, the corresponding velocity v_j will be approximated by (8) or (9).

In order to get the right sign for the volume flow (positive for blow notes and negative for draw notes) one has to take care of the sign of the reed openings h_j. Reed openings are positive for the draw reed ($j = 2$), but they are negative for the blow reed ($j = 1$). Using $(-1)^j\, h_j$ instead of h_j will give the right (positive) sign for blow and draw reeds.

The cross-sectional area A for the volume flow through the gaps between reed and reedplate and its influence on the dynamical behaviours of the reed is discussed in great detail in [MB 07]. In this essay, however, the overall cross-sectional area will simply be approximated by $A = (-1)^j\, W h_j$ with an effective width W of the reed, e.g. 2 times the measured width.

Pumped flows are proportional to reed surface S and reed velocity $\dot{h}_j$. S is an effective value which equals the effective reed surface used in (1) for calculating the pressure force on the reed [MB 07] . For both blow and draw notes, the pumped flow directed towards the outside counts positive. Both reeds have positive velocity $\dot{h}_j$ on their way to the outside, therefore both reeds contribute $+S\dot{h}_j$ to the overall volume flow.

Summing up, we have

$$Q_j = (-1)^j\, W h_j v_j + S \cdot \dot{h}_j$$

A linear approximation of the volume flow $Q_j\left(h_j, v, \dot{h}_j\right) = (-1)^j\, W h_j v_j + S \cdot \dot{h}_j$ around the point $Q\left(h_{j0}, v_{j0}, 0\right)$ is obtained by partial derivation :

$$Q_j\left(h_j, v_j, \dot{h}_j\right) = Q_j\left(h_0, v_0, 0\right) + (-1)^j \left(\frac{\partial Q_j}{\partial h_j}|_{h_{j0}} \cdot (h_j - h_{j0}) + \frac{\partial Q_j}{\partial v_j}|_{v_{j0}} \cdot (v_j - v_{j0})\right) + \frac{\partial Q_j}{\partial \dot{h}_j}|_0 \cdot \dot{h}_j$$

With $h_j = h_{j0} + \hat{h}_j \cdot \exp i\omega t$, $\dot{h}_j = i\omega\hat{h}_j \cdot \exp i\omega t$ and $v_j = v_{j0} + \hat{v}_j \cdot \exp i\omega t$ it follows (ignoring terms proportional to $\exp \omega t \cdot \exp \omega t = \exp 2\omega t$):

$$Q_j(\omega) = W\,(-1)^j \left(h_{j0} \cdot v_{j0} + v_{j0}\hat{h}_j \cdot \exp i\omega t + h_{j0}\hat{v}_j \cdot \exp i\omega t\right) + S i\omega\hat{h}_j \cdot \exp i\omega t$$

Therefore the volume flow can be written as $Q_j = Q_{j0} + \hat{Q}_j \exp i\omega t$ with:

$$Q_{j0} = W\left(-1\right)^j v_{j0} h_{j0} \tag{12}$$

$$\hat{Q}_j = W\left(-1\right)^j \left(v_{j0}\hat{h_j} + h_{j0}\hat{v_j}\right) + Si\omega \hat{h}_j \tag{13}$$

Inserting $\hat{h}_j$ from (4) and $\hat{v_j}$ from (11) into (13) gives:

$$\hat{Q}_j = \left(\left(-1\right)^j \cdot W \left(\frac{v_{j0} \cdot S}{-\omega^2 m_j + i\omega\gamma_j + K_j} + h_{j0}\sqrt{\frac{2}{\varrho}} \cdot \frac{1}{2\sqrt{|p_0|}} \right) + \frac{i \cdot S\omega \cdot S}{-\omega^2 m_j + i\omega\gamma_j + K_j} \right) \cdot \hat{p} \tag{14}$$

Single reed admittanc

Equation (14) describes the linear approximation of a nonlinear single reed system in frequency space and thus defines an input admittance[3] $Y_j\left(\omega\right) = \hat{Q}_j/\hat{p}$. Using (3) and (10) will bring input pressure (an externally controllable variable) into play (positive for blow notes and negative for draw notes). In addition, it is more common in literature [Sil 09, ChK 16] to apply $\omega_{j0}^2 = K_j/m_j$ (with reed mass m_j and reed eigen frequency ω_{j0}) and use the damping constant q_j, which is defined by $\gamma_j = m_j\omega_{j0}q_j$ (its inverse $1/q_j$ is the so-called quality factor Q of an oscillator):

$$\frac{S}{K_j - \omega^2 m_j + i\omega\gamma_j} = \frac{S}{m_j\omega_{j0}^2 - \omega^2 m_j + i\omega_{j0}\omega m_j q_j} = \frac{S}{m_j} \cdot \frac{1}{\omega_{j0}^2 - \omega^2 + i\omega_{j0}\omega q_j}$$

So we finally have for the input admittance of a single blues harp reed:

$$Y_j\left(\omega\right) = W\sqrt{\frac{2\left|p_0\right|}{\varrho}} \cdot \left(-1\right)^j \left(\pm\frac{S}{m_j} \cdot \frac{1}{\omega_{j0}^2 - \omega^2 + i\omega_{j0}\omega q_j} + \frac{h_{j,gap} + Sp_0/\omega_{j0}^2 m_j}{2\left|p_0\right|} \right) + \frac{S^2}{m_j}\frac{i \cdot \omega}{\omega_{j0}^2 - \omega^2 + i\omega_{j0}\omega q_j} \tag{15}$$

As before, $j = 1$ stands for the blow reed, $j = 2$ for the draw reed. The upper sign in $\pm$ is valid for blow notes, the lower sign for draw notes, due to the formula (10) for v_{j0}.

Blues harp and clarinet

Formula (5.66) in [RH 04] describes the flow from the mouthpiece of a clarinet into the pipe (not taking care, however, of the pumped flow). Solving it for the input admittance Y_p of the pipe and switching the sign of the resulting formula yields the input admittance $Y_r = -Y_p$ of the reed system seen from the boundary between mouthpiece and pipe. The result is a formula for Y_r of a clarinet reed, equivalent to our formula (15) for a blues harp reed (neglecting, of course, the last summand coming from the pumped flow) applied to a *draw* note on a *draw* reed (because of $h_{j0} > 0$ for draw reeds).

But this is exactly what one should expect. The reason is the double role played by the vocal tract of a blues harp player. There is some mean pressure inside the mouth driving the self-excited feedback system, and the vocal tract acts as resonator. One can separate the two roles by shifting the pressure axis. If the pressure for a *draw* note inside the vocal tract equals $p = p_0 + \hat{p} \cdot \exp i\omega t$ and the pressure outside the instrument is approximately equal to atmospheric pressure, subtracting $p_0 < 0$ gives a constant positive pressure $-p_0 > 0$ outside the mouthpiece (now thinking of a clarinet) and pressure fluctuations $\hat{p} \cdot \exp i\omega t$ inside the pipe. Clarinet reeds act as closing reeds, and this property will not be changed by the mathematical operation of shifting the pressure axis. Therefore the equivalent blues harp reed will be a closing reed too. For a draw note this is a draw reed, and we are done with our justification.

Summing up, formula (15) for the admittance of a blues harp reed is essentially equivalent to formulas found in literature for woodwind instruments [RH 04, ChK 08, ChK 16, FaBe 12, Fle 79] (see also [Fö-F, Fö-J]).

[3] [RH 04, 5.65d] "impedance $Z_p\left(\omega\right)$" is also used for arbitrary complex values of ω. Looking for conditions of instability, ω will be real, thus "impedance" or "admittance" will be understood as usual.

Admittance of the blues harp

The volume flow at the boundary between vocal tract and reed channel equals the sum of the volume flows through the two reed gaps. Assuming homogeneous pressure the input admittance $Y(\omega)$ of the reed channel thus equals the sum of the admittances of the single reeds:

$$Y(\omega) = \sum_{j=1,2} W\sqrt{\frac{2|p_0|}{\varrho}} \cdot (-1)^j \left(\pm \frac{S}{m_j} \cdot \frac{1}{\omega_{j0}^2 - \omega^2 + i\omega_{j0}\omega q_j} + \frac{h_{j,gap} + Sp_0/\omega_{j0}^2 m_j}{2|p_0|} \right) + \frac{S^2}{m_j} \frac{i \cdot \omega}{\omega_{j0}^2 - \omega^2 + i\omega_{j0}\omega q_j} \tag{16}$$

In order to write down the real and the imaginary part as well as magnitude and phase shift we will use the denominator D:

$$D = \left(\omega_{j0}^2 - \omega^2\right)^2 + \left(\omega_{j0}\omega q_j\right)^2 \tag{17}$$

$$Y(\omega) = \sum_{j=1,2} W\sqrt{\frac{2|p_0|}{\varrho}} \cdot (-1)^j \left(\pm \frac{S}{m_j} \cdot \frac{\omega_{j0}^2 - \omega^2 - i\omega_{j0}\omega q_j}{D} + \frac{h_{j,gap} + Sp_0/\omega_{j0}^2 m_j}{2|p_0|} \right) + i\omega \frac{S^2}{m} \cdot \frac{\omega_{j0}^2 - \omega^2 - i\omega_{j0}\omega q_j}{D} \tag{18}$$

Remember that the positive sign in $\pm$ stands for blow notes and the negative sign for draw notes. The real and the imaginary parts of the admittance of a blues harmonica are:

$$\mathrm{Re}Y(\omega) = \sum_{j=1,2} W\sqrt{\frac{2|p_0|}{\varrho}} \cdot (-1)^j \left(\pm \frac{S}{m_j} \cdot \frac{\omega_{j0}^2 - \omega^2}{D} + \frac{h_{j,gap} + Sp_0/\omega_{j0}^2 m_j}{2|p_0|} \right) + \frac{S^2}{m_j}\omega^2 \cdot \frac{\omega_{j0} q_j}{D} \tag{19}$$

$$\mathrm{Im}Y(\omega) = \sum_{j=1,2} \pm W\sqrt{\frac{2|p_0|}{\varrho}} \cdot (-1)^{j+1} \cdot \frac{S}{m_j} \cdot \frac{\omega_{j0}\omega q_j}{D} + \frac{S^2}{m_j}\omega \cdot \frac{\omega_{j0}^2 - \omega^2}{D} \tag{20}$$

3 Channel #4 of a C-harp

Measurements

For channel #4 of a C-harp (Hohner Special 20 MARINE BAND) I measured and calculated the values in the table (see the Appendix and [Fö-R]):

	blow reed	draw reed
S	$1.1 \cdot 10^{-5} m^2$	$1.1 \cdot 10^{-5} m^2$
ω_{j0}	$3300 s - 1$	$3700 s^{-1}$
q	0.004	0.004
m_j	$3.9 \cdot 10^{-6} kg$	$3.1 \cdot 10^{-6} kg$
W	$2 \cdot 0.002 m$	$2 \cdot 0.002 m$
$h_{j,gap}$	$-0.2 \cdot 10^{-3} m$	$0.2 \cdot 10^{-3} m$

Contributions to volume flow for single reeds

Fig. 4 resp. Fig. 5 show the real and imaginary parts of the input admittance of a single blow reed (left) and a single draw reed (right) for a blow note resp. for a draw note[4]. The real parts are shown in red (dashed), the components are shown in purple (oscillating reed opening, mean pressure p_0), orange (oscillating flow through mean opening h_{10}) and pink (pumped flow). The imaginary parts are shown in blue (dashed), the components are shown in green (oscillating reed opening, mean pressure p_0) and light blue (pumped flow).

All four plots show nearly identical red and purple curves. The real part of the input admittance of the reeds is obiously exclusively determined by the dynamic answers of the two reeds. For the imaginary part, the pumped flow is equally important. Velocity fluctuations inside the mean reed openings may be neglected within harmonic stability analysis.

[4]All plots in this paper were created using open GeoGebra [GG] (open source freeware)

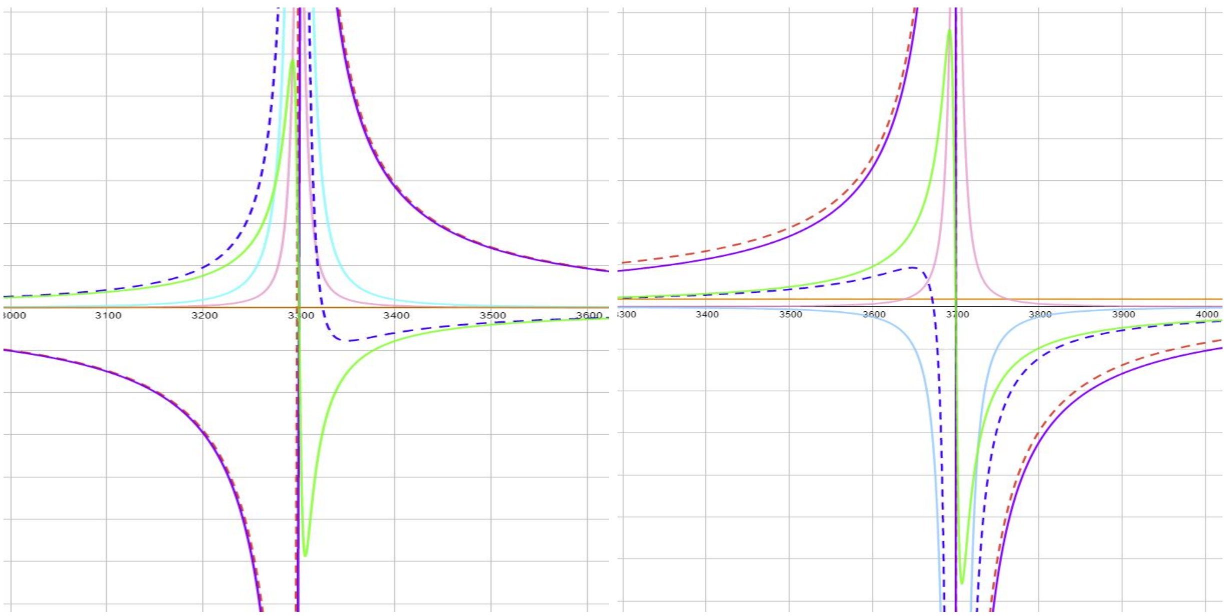

Figure 4: Real and imaginary parts of the input admittance of a single blow reed (left) and a single draw reed (right) for a blow note.

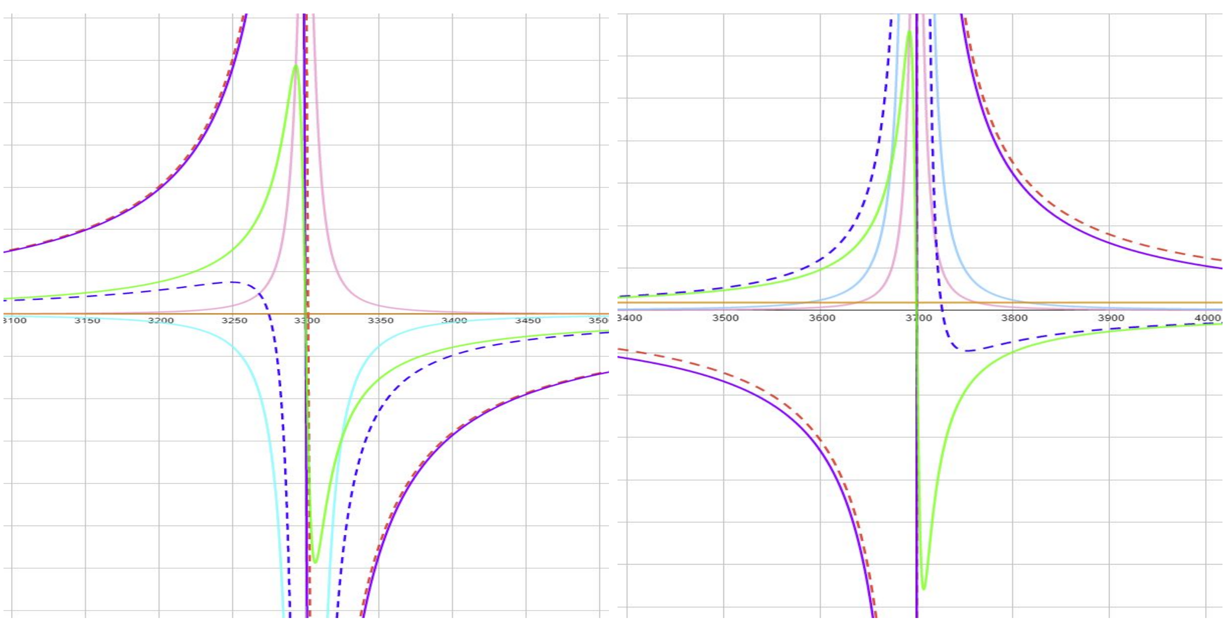

Figure 5: Real and imaginary parts of the input admittance of a single blow reed (left) and a single draw reed (right) for a draw note.

Vocal tract resonances

In contrast to the saxophone, the blues harmonica has no resonator volume of its own[5]. However, a reasonable candidate for a resonator should be the player's respiratory tract together with the reed channel. Referring once more to the saxophone, the influence of the embouchure on playing frequency can be modelled within the usual resonator model by treating vocal tract and bore impedances in series [CSW 11]. Subjective experience of the author shows that one can bend notes on a tenor saxophone with configurations of the vocal tract used for bending notes on the harmonica.

[5]For very high notes, the volume of the reed chamber could come into play.

MRI images [ESBRH 13] confirmed that the player's tongue position has a significant influence on the formation of bends and overbends. Most important seems to be a constriction between the tongue and the palate, which for high tones (up[6] to about Dis 6 $= 1245Hz$) is located far forward in the mouth as with the vowel [y], and for low tones (down to about G3 $= 196Hz$) is located far back in the throat as with the vowel [u]. Tones in between are playable accordingly with tongue positions in between.

Vowels in speech production are characterized by *formants*, and formants are caused by vocal tract resonances. Following Fant [F70], the constriction between tongue and palate defines a back cavity and a front cavity volume. Together with the lips channel, this constriction and the various volumes give rise to different kinds of standing waves and Helmholtz resonances, which can explain the observed formant frequencies. The evaluation of MRI images confirmed this intuititive approach [SMWS 92] which is refined by calculation of formants with the help of transmission line techniques [K 00, MSBA 07].

The tongue positions described above also occur when whistling. *Whistling* can be explained as self-excited oscillation with the combination of front cavity and lip channel as a Helmholtz resonator [F70, APM 18]. The amplification of the high pitch of the biphonic sound in *overtone singing* is caused by the coincidence of resonances and formants [KNRS 01, KNR 03, ENS 20].

Quiescent medium inside the vocal tract?

The equations of linear acoustics including the definitions of an input admittance or an impedance assume a quiescent medium. But of course there is a constant air flow superimposed to the acoustic fluctuations inside the vocal tract. Otherwise, the lungs would neither fill with air nor would they be emptied.

How fast is the air inside my vocal tract when playing e.g. a bend note on channel #4 of a C-harp? I can hold such a note for about $30s$, my lung volume is about $4l$, the sectional area of the front part of my vocal tract is about $5cm \cdot 5cm$. With these values the mean velocity inside my mouth is about $0.5m/s$, the velocity of very slowly walking. Therefore the constant flow inside the vocal tract can be neglected and vocal tract admittance at the boundary to the reeds system is well-defined.

In contrast, the mean velocity of air flowing through the palatal constriction cannot be neglected. The respective air flow may be modelled e.g. by a Bernoulli equation, this time neglecting acoustic waves.

Single mode resonator

To my knowledge, there are neither measurements nor model calculations for the admittance of the vocal tract while playing the blues harp. As a toy model we will use the first summand of the impedance mode expansion found in [ChK 08, ChK 16]:

$$Z_{res}\left(\omega\right) = i\omega \sum_{n} \frac{F_n}{\omega_n^2 - \omega^2 + i\omega\omega_n q_n} \tag{21}$$

The expansion uses resonance frequencies ω_n, damping factors q_n and modal factors F_n. With $n = 1$ we get the admittance $Y_{res}\left(\omega\right)$ of a single mode resonator:

$$\begin{aligned} Y_{res}\left(\omega\right) &= \frac{\omega_1^2 - \omega^2 + i\omega\omega_1 q_1}{i\omega F_1} \\ Y_{res} &= \frac{q_1\omega_1}{F_1} + i\frac{\omega^2 - \omega_1^2}{\omega F_1} \end{aligned} \tag{22}$$

Linear stability analysis

Assuming continuity of pressure and volume flow at the boundary between reeds system and resonator, minus the input impedance of the reeds system will equal the input impedance of the resonator: $-Y = Y_{res}$. There has to be one minus sign, because both input impedances are based on flow axes showing to the inside of the respective system.

[6] In former times you could buy blues harmonica from G (lowest tuning) in half steps up to Fis (highest tuning). D 6 was thus the highest attainable draw bend on channel #6 of a Fis-harmonica. There were also available extra tunings in "low C". G3 on channel #2 of such a harp is the lowest note the author is able to bend.

In playing practice, one choses ones embouchure as well as pressure p_0 unconsciously in "the right way" to play the desired note. Linear stability analysis means (following [RH 04]) determining threshold pressure p_0 and playing frequency ω (the frequency of the most unstable mode) by solving $Y_{res} = -Y$ for given resonator admittances (18). In contrast, most of the examples below are results of first fixing mode frequency ω_1 and pressure p_0 and then "playing" with damping factor q_1 and modal factor F_1 until $-Y(\omega) = Y(\omega)_{res}$ is visible for some playing frequency ω.

In all examples, both the real and imaginary parts of $-Y$ and of Y_{res} will be plotted in order to look for frequencies ω with common real resp. imaginary parts of $-Y$ and of Y_{res} as candidates for possible playing frequencies.

"Negative energy"

Because the real part of the resonator admittance (22) is always positive, solutions of linear stability analysis can only exist for positive real parts of $-Y$, i.e. for negative real parts of the input admittance of the resonator. Thus the "negative resistance condition" [Fle 79, Jo 87] for playing frequencies is justified without referring to "negative energy" in (rough) analogy to electric AC circuits.

Equal phase shifts: necessary, but not sufficient

If both real and imaginary parts of $-Y$ (with Y the reeds admittance) and resonator admittance Y_{res} are equal, phase angles of the corresponding complex numbers have to be equal too. Looking for playing frequencies within our toy model, it will therefore be helpful to plot the curve belonging to (23). Zeros of this curve are candidates for playing frequencies:

$$y = \arctan \frac{ImY}{ReY} - \arctan \frac{\left(\omega^2 - \omega_1^2\right)/\omega}{q_1\omega_1} \tag{23}$$

But, as a warning, not all zeros of (23) are possible playing frequencies.

Fig. 6 shows the real (red) and imaginary (blue) part of $-Y$ for a blow note on channel #4 of a C-harp as well as the real (green) and imaginary (orange) part of the admittance Y_r of a one mode resonater with resonance frequency $3500s^{-1}$ (the zero of ImY_r) , lying between the eigen frequences $3300s^{-1}$ and $3700s^{-1}$ of the two reeds. The parameters used here and in the figures below are listed in the Appendix.

The nearly vertical lines (violett) in Fig. 6 are plots of the function (23) for a blow note on channel #4 of a C-harp. The zero between $3300s^{-1}$ and $3700s^{-1}$ belongs to a negative part of $-Y$ and can therefore be no playing frequency.

But also the four zeros of the vertical lines with positive real parts of $-Y$ do not represent possible playing frequencies. The respective complex numbers $-Y$ and Y_{res} have the same phase angle, but they have obviously different real and imaginary parts, i.e. they are different (they have different magnitude).

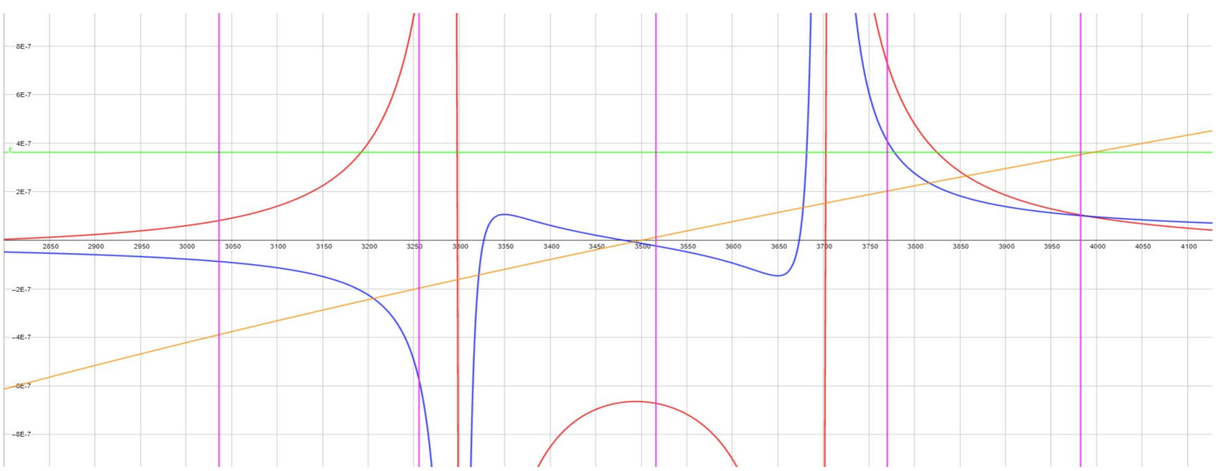

Figure 6: Artifacts: common phase angles, but no playing frequency allowed.

Drawbend on channel #4

Fig. 7 shows two ways to get a drawbend. A mode frequency of $3600s^{-1}$ between the eigen frequencies of the reeds results in a drawbend with frequency $3535s^{-1}$ below mode frequency. A mode frequency of $3200s^{-1}$ below the eigen frequencies of both reeds results in a playing frequency $3368s^{-1}$ above mode frequency.

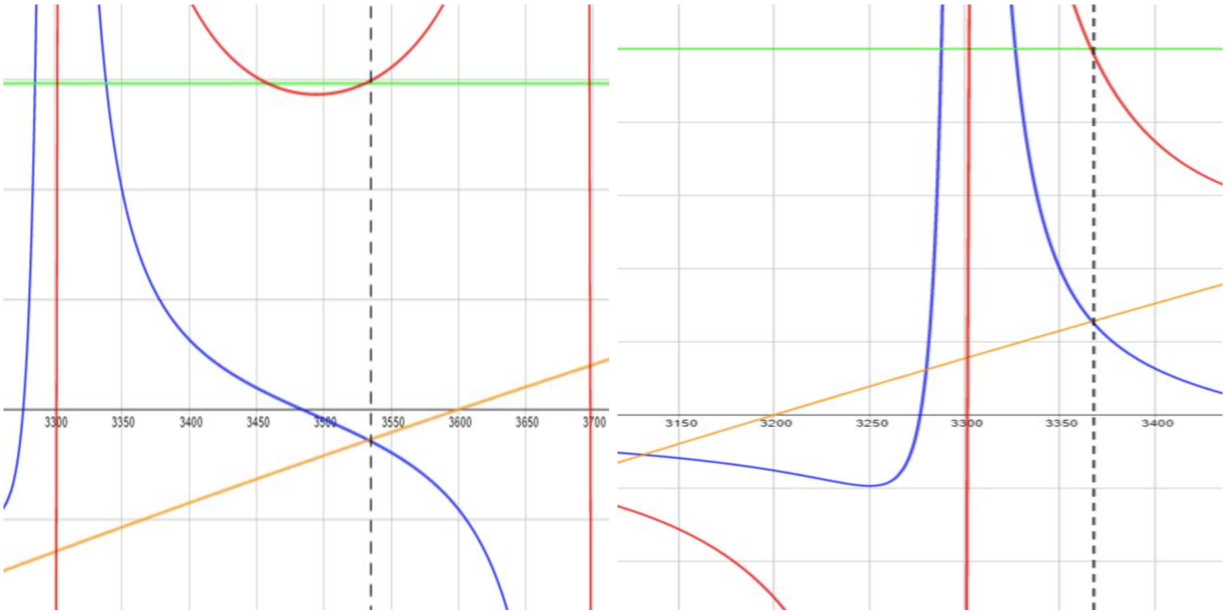

Figure 7: Drawbends

Normal notes

Normal notes can be played with relaxed embouchure, so no resonator properties of the vocal tract seem to be involved. Playing normal notes, resonance properties influence tone quality, but probably not the existence or non-existence of self-excited oscillations. If anything, a generator resonator model will only be able to explain bend or overbend notes. As it is possible to bend normal notes downward seamlessly and continuously, one should look for a higher level model able to explain both kinds of notes.

Blowbend and overblow on channel #4

Next we will consider blow notes. Fig. 8 (left) with resonator frequency $\omega_1 = 3250s^{-1}$ predicts instabilities of the combined system of reeds and vocal tract for a playing frequency $3221s^{-1}$ below the pitch $3300s^{-1}$ of the blow reed. This would be a *blowbend*. Such instabilities actually exist in playing practice. It is, however, impossible to play blowbends on the lower channels loudly and with good tone. This is no contradiction - linear stability analysis says nothing about the fate of perturbations at long times [Fie, 6.6].

In contrast, Fig. 8 (right) with $\omega_1 = 3800s^{-1}$ shows a playing frequency of $3888s^{-1}$, lying the eigen frequency $3700s^{-1}$ of the draw reed. Such a note can actually appear in playing practice as a *(bended) overblow*.

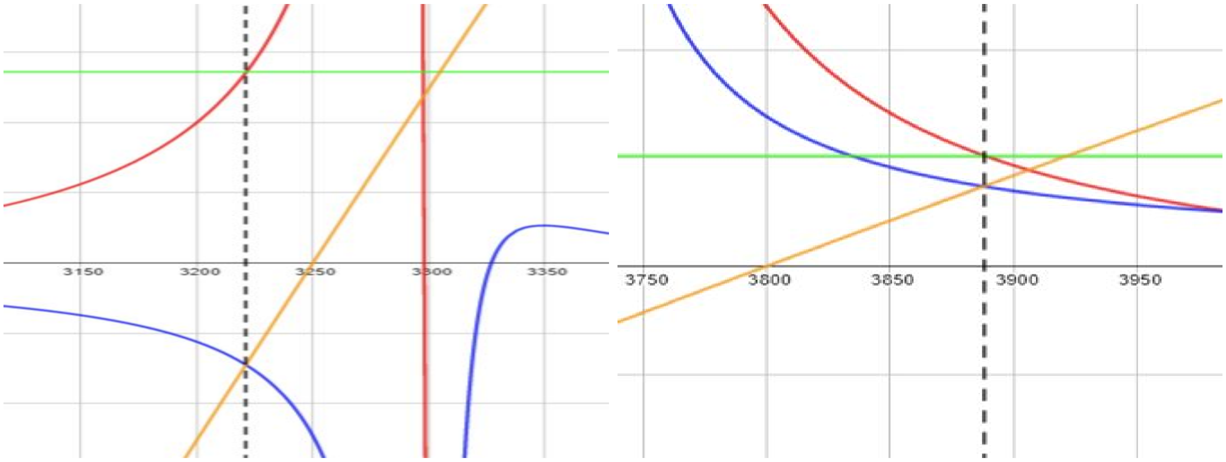

Figure 8: Left: Blowbend. Right: Overblow

Drawbend and overblow

Fig. 9 predicts drawbend and overblow for identic values of resonator frequency of $3500s^{-1}$ between the pitches of the two reeds, equal q_1 and for a value F_1 about twice as large for the blowbend. Mean pressures p_0 are $-700Pa$ for the drawbend and $1310Pa$ for the overblow.

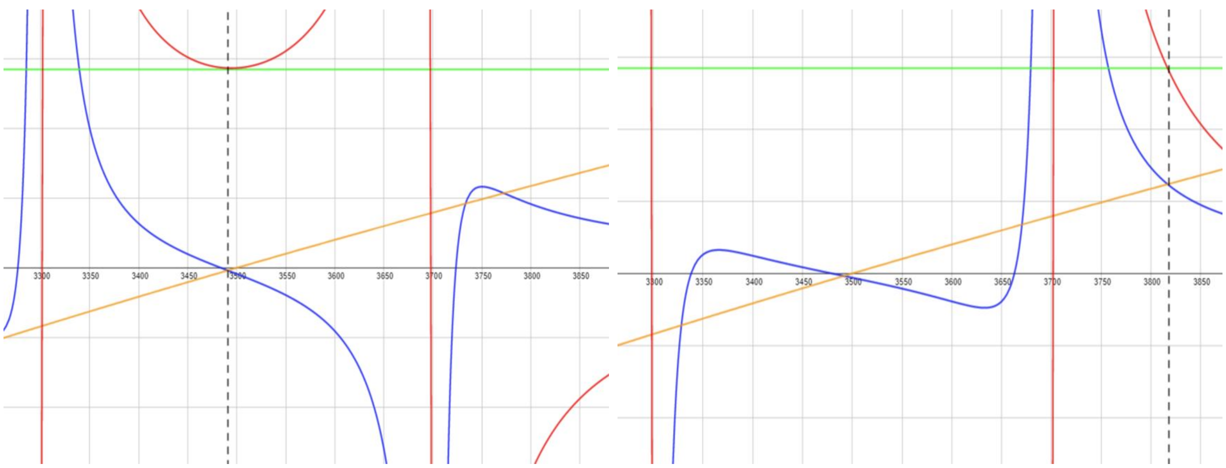

Figure 9: Drawbend and overblow with common resonator frequency

4 Discussion

The onset of oscillations in experiment

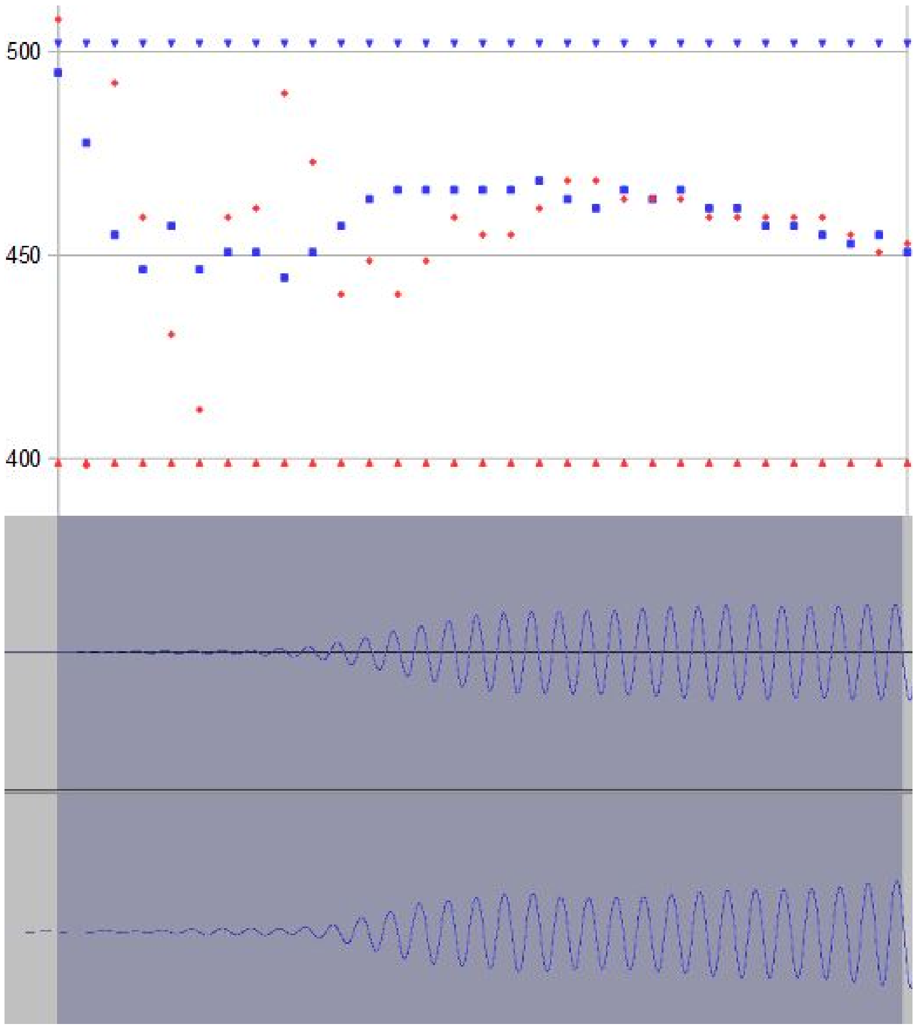

Figure 10: Blues harp reeds in channel #3 of a C-harp start oscillating (see the Appendix "Measurements"). Above: frequencies determined over a period length. Red: blow reed, blue: draw reed. Below you can see the wave file in linear representation synchronously to the obtained frequencies (top: draw reed, bottom: blow reed). The blow reed has a natural frequency of $399Hz$, the draw reed a natural frequency of $502Hz$.

One should be aware of the limits of the modelling approach introduced above. Linear stability analysis is built on (as the name says) a linear approximation of a non-linear system. It offers *necessary* conditions for very small sinusoidal oscillations of the interacting system of reeds and air flow, but not more. There is no guarantee that the oscillations will increase [Fie], and there is also no explanation, why such small sinusoidal oscillations with common frequency should build up at all at the very beginning of blowing or drawing.

Fig. 10 shows experimental results, starting playing a drawbend on channel #3 of a C-harp (see [Fö-T] for more detailed informations). Frequencies f are calculated as $f = 1/T$ by estimating lengths T of single oscillations of the two reeds. At the beginning there are no oscillations with defined frequency. While the amplitudes increase, the frequencies for the two reeds stabilize. After about five hundredths of a second, both reeds oscillate with the same frequency, but already in the nonlinear saturation range.

More than a toy model?

Modelling the player's vocal tract (or, to be more precise, the air flow between lungs and reed gaps) by a *single* mode resonator with arbitrary chosen parameters (damping factor and modal factor) might be not more than a toy model. Vocal tract configurations used for bends or overbends are somewhat similar to geometries used for pronouncing vocals like [i] or [u], which are related to *several* resonances (formants). On the other hand, free and weakly damped reeds obey harmonic oscillator equations and should be mainly influenced by pressure oscillation frequencies not too far away from their eigen frequencies. Therefore, a *single* resonator frequency might be a good approximation.

As mentioned above, the palatal constriction in the vocal tract is crucial for playing bends and overbends. This constriction together with the anterior oral cavity defines a Helmholtz resonator. [BAB98] measured values of the cavity volume of a professional player (Howard Levy) when playing half-tone draw bends on various harmonicas. In [Fö-R] these volume values together with values for length and area of the constriction of another player (David Barrett) playing an half-tone bend [ESBRH 13] are plugged into the Helmholtz resonator formula. The calculated resonance frequencies are in quite good agreement with the respective playing frequencies in [BAB98]. This result likewise suggests that the vocal tract could be modeled by a single-mode resonator.

Drawbends

As in [Jo 87], a necessary condition for draw notes on channel #4 is a playing frequency between the eigen frequencies of the two reeds. A necessary condition for blow notes is, that playing frequency lies below the eigen frequency of the blow reed or above the eigen frequency of the draw reed. Differing from [Jo 87], we maintain that a resonator theory cannot explain normal notes.

In playing practice, frequencies of drawbends are indeed limited to the interval between both eigenfrequencies. The single mode model predicts playing frequencies of drawbends not far away from resonator frequency (see Fig. 7) and Fig. 9), which seems plausible (for a discussion of experiments in [BAB98] or [Jo 87] see [Fö-R]). There is a multitude of vocal tract configurations for playing a desired drawbend. Different parameters for one fixed playing frequency are therefore no contradiction. Starting with some vocal tract configuration (parameter setting), smooth changes of this configuration (parameter setting) result in smooth changes of resonator frequency and playing frequency: playing practice and model fit together.

It is, however, possible to bend on two or three neighbouring channels simultaneously or to bend octaves (using tongue block for the channels between). It is further possible to play slides consisting of bend notes with fixed ambouchure [Fö-M]. Here, a simple resonator model will certainly reach its limits.

Blowbend versus overblow

Fig. 8 predicts blowbends and overblows on channel #4. It is indeed possible to play blowbends with low volume, and linear stability analysis is about small osciallations. It is furthermore possible to play overblows. Following those figures, resonator frequencies below resp. above blow reed pitch are required for playing blowbends resp. overblows.

If one actually tries to bend a normal blow note down, a blowbend will "pop up" instead. Fig. 11 shows wave form (linear) and frequencies for such an experiment. The key here is that I focused on keeping the shape of my vocal tract constant while playing. It should therefore be possible to model (in contrast to Fig. 8) a low volume blowbend and an overblow with *common* resonator frequency (but so far I have not succeeded in doing so).

The crucial question of why it is impossible to play loud blowbends and why the overblow always "wins" will probably be beyond the reach of linear stability theory.

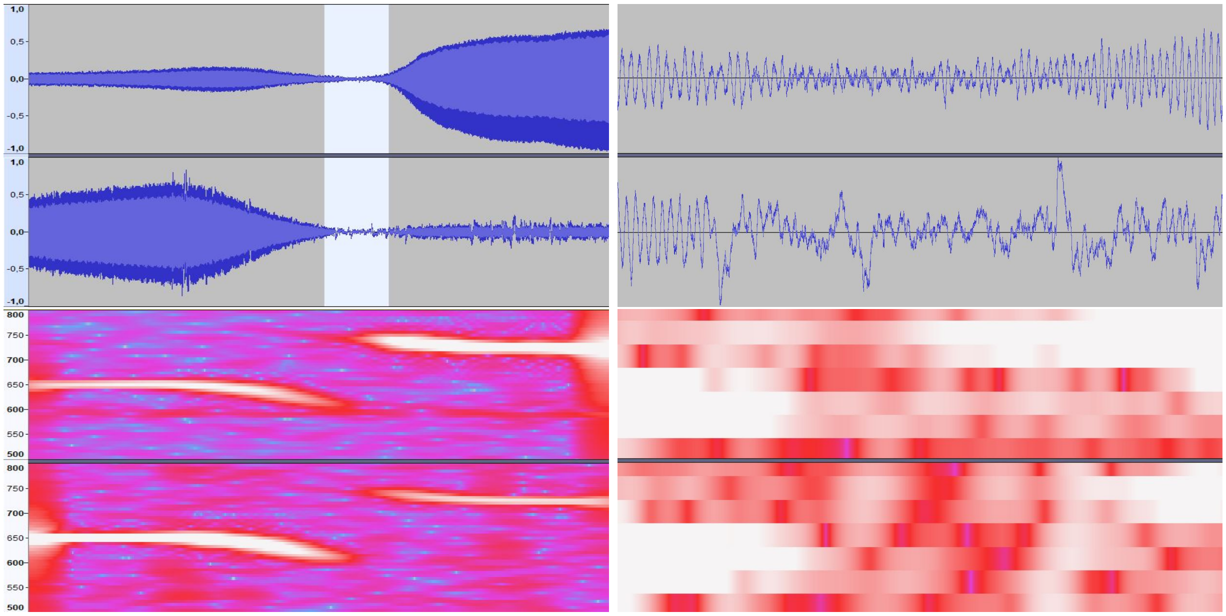

Figure 11: Normal blow note, followed by blowbend "popping up" to overblow (see the Appendix "Measurements"). For the plots on the right, the interval marked on the left was amplified and stretched in time.

Switching from drawbend to overblow

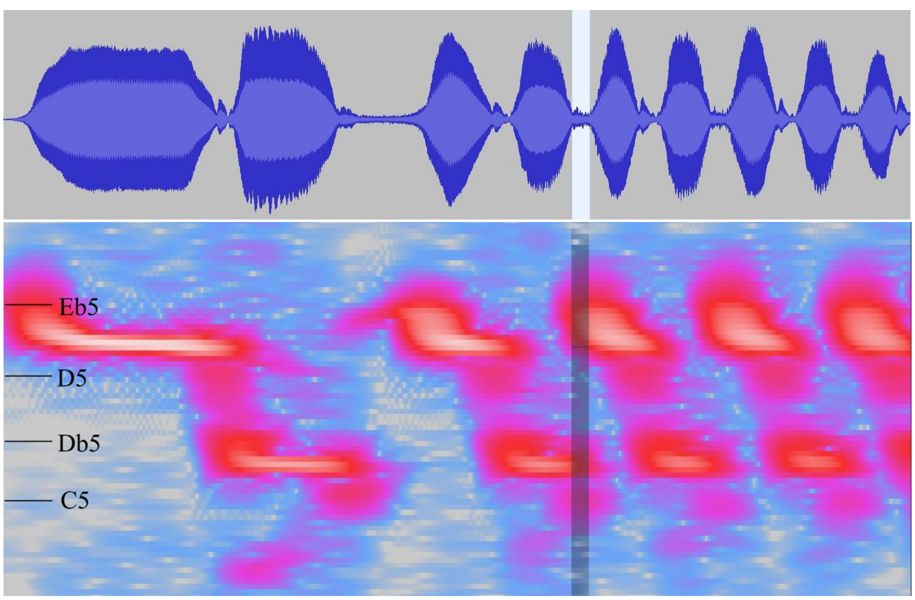

Figure 12: Switching between overblow and draw bend with fixed embouchure (see the Appendix "Measurements"). Wave in sync with spectrogram.

In contrast, Fig. 9 fits playing experience. It is possible to switch between drawbends and overblows, thereby fixing one's vocal tract configuration. The wave diagramm and the corresponing spectrogram in Fig. 12 show overblows and draw bends on channel #4 of a C-harp with frequencies $606Hz$ (overblow) resp. $545Hz$ (draw bend). The series of notes evaluated in the right half of the figure was obtained by rapidly switching between inhaling and exhaling while keeping one's embouchure fixed. Before and after the marked rectangle one hears clearly a draw bend resp. an overblow of constant playing frequency. The rectangle marks a switching time of ca. $0.07s$. It seems very unlikely that the player's embouchure, together with the corresponding resonator frequency, has changed significantly during this short time interval.

Appendix

Parameters used in figures

Fig.	6	7 (left)	7 (right)	8 (left)	8 (right)	9 (left)	9 (right)
p_0/Pa	700	-700	-700	700	700	-700	1310
ω_1/s^{-1}	3500	3200	3600	3250	3800	3500	3500
$F_1/10^{10}Pa/m^3$	0.2569	$0,1275$	0.242	0.02	0.1174	0.3967	0.245
q_1	0.268	0.399	0.349	0.0335	0.063	0.399	0.399

Measurements

Reed parameters

Own measurements of the reed parameters were made for channel #4 of a customized Hohner Special 20 Marine Band in C (standard tuning).

Spring constant and relaxation time were measured exemplarily for one of the two reeds (blow reed resp. draw reed) in channel #4 of a C-harp. These values should also apply to the other reed within the scope of the rounding and the usual manufacturing tolerances.

A load of $4g$ at the free end of the C-reed causes a deflection of $1mm$, resulting in a spring constant of $42N/m$ with $g = 9.8N/kg$ and (26). Due to the weak damping, the resonant frequencies of the reeds are practically equal to the (pseudo) eigenfrequencies measured during the plucking: $f_0 = 589Hz$ (D-reed) and $f_0 = 525Hz$ (C-reed) with the corresponding angular frequencies $\omega_0 = 3700s^{-1}$ and $\omega_0 = 3300s^{-1}$. For the C-reed, this gives an effective mass of $3.9mg$ using (29), and for the D-reed we get $3.1mg$. In comparison, the actual mass of the oscillating part of the C-reed is $42mg$, which illustrates the necessity of using effective quantities when modeling a vibrating reed by an abstract 1-point oscillator.

For the pressure force, the effective surface area S is only $2/5$ of the actual surface area, thus length $14mm$ and width $2mm$ results in $S = 1.1 \cdot 10^{-5}m^2$. The clearance gap is $x_{gap} = \pm 2 \cdot 10^{-4}m$, the damping constant is $\chi = 0.004$ (measured for the D-reed via the relaxation time).

Realistic blowing pressures during bending are $p_0 = -700Pa$ according to [Mi 99].

Reed oscillations

A Hohner Special 20 Marine Band was inserted into a TurboHarp ELX [ELX] by James Antaki. In the ELX the reed oscillations are converted into electrical voltage fluctuations with the help of optical sensors. The signals coming from the blow reeds or draw reeds can be monitored separately via a stereo output. The ELX acts as a "black box" with unknown transformation properties concerning absolute values of elongations or exact values of phase angles between the movements of the two reeds. One can, however, deduce information about periodicity as well as approximate information about the shape of the oscillations.

The stereo signal of the *ELX* was recorded with a *Zoom H4n* mobile phone recorder with a sampling frequency of $96kHz$ and a resolution of $24Bit$. The recordings were made without power devices and artificial lighting, so that the recording device and optical sensors were not disturbed by a $50Hz$ or $100Hz$ signal (Fig.10 and Fig.11). Emitted sound was recorded with the build-in mics of the *Zoom H4n* (Fig.12).

The wave files were evaluated using the audio software *Audacity* (*free* und *open source*) [Aud]. Fig.10, Fig.11 and Fig.12 show waveforms in linear representations. The spectrograms in Fig. 11 (left) and Fig.12 are calculated with a Hann window (16384 samples), in Fig.11 (right) the wave was amplified and stretched in time and a Hann window (2048 samples) was used.

Oscillating reed and equivalent 1-point oscillator

Effective quantities

A reed oscillating under the influence of external pressure is, from a physical point of view, a spatial many-body problem. The simplest oscillation mode can be described by the equation of motion of a 1-point oscillator:

$$\ddot{x} + 2r\dot{x} + \omega_0^2 x = \frac{S \cdot p}{m} \tag{24}$$

$$\omega_0^2 = \frac{k}{m} \tag{25}$$

All quantities in the equivalent oscillation equation (24) are to be regarded as effective (formal) quantities, whose relation to the real, experimentally accessible reed oscillation shall be established in the following.

- x is identified with the deflection of the tip of the reed from its rest position.
- It is assumed that the pressure difference between both sides of the reed surface has approximately a constant value p across the entire reed, so that the effective force on the right side of (24) can be calculated as $S \cdot p$ by means of an effective surface S. According to [Mi 99, Annexe A] S is equal to $\frac{2}{5}$ of the oscillating surface of the reed.
- ω_0 is the effective resonance frequency, which must be distinguished from the effective natural frequency.
- r is an effective damping constant, m is the effective mass, k is the effective spring constant (spring stiffness).

It is easy to measure the deflection x of the reed tip under the influence of a force F_S which is applied to the tip. This results in a spring constant (spring stiffness) $k_S = F_S/x$. According to [Mi 99, Annexe A] the effective spring constant in (25) is then equal to

$$k = \frac{16}{15} k_S \tag{26}$$

By plucking the reed, the natural frequency and relaxation time of the reed can be determined. These experimentally accessible quantities should be directly transferable to the equivalent oscillator. If the description of the real oscillating reed by the fictitious 1-point oscillator is reasonable, the effective damping constant r , the effective resonant frequency ω_0 (which should agree with the actual resonance frequency which can also be measured) as well as an effective mass m (which has no simple relation to the real oscillating mass) can be determined:

Relaxation time and damping

The *damping coefficient* r of a reed and its *natural frequency* ω_0 can be determined by plucking ($x_0 > 0$ and $v_0 = 0$) and measuring the relaxation time τ and the occurring frequency ω.
While the reed is oscillating freely, no external force is acting. Thus we have for the elongation x:

$$\ddot{x} + 2r\dot{x} + \omega_0^2 x = 0$$

The solution is $x = x_0 \cdot e^{-r \cdot t} \cdot \cos(\omega t - \varphi)$ with *eigenfrequency* $\omega_E = \sqrt{\omega_0^2 + r^2}$ and $\varphi = \arctan \frac{r}{\omega}$.
Relaxation time τ and *damping coefficient (damping constant)* r are therefore reciprocal to each other (which motivates the factor2 in the oscillator equation):

$$r = \frac{1}{\tau} \tag{27}$$

Eigenfrequency and effective mass

From the measured *natural frequency* $\omega_E = \sqrt{\omega_0^2 + r^2}$ results the *resonance frequency*:

$$\omega_0 = \sqrt{\omega_E^2 + r^2} \tag{28}$$

The damping constant and the resonance frequency of the reed can therefore be taken from the decay curve. Since r is much smaller than ω for blues harp reeds, there is practically no difference between the resonant frequency of the reed and the frequency heard after plucking.
Together with (26) this gives the effective mass of the 1-point oscillator:

$$m = \frac{k}{\omega_0^2} \tag{29}$$

Damping and quality factor

Quality factor Q, relaxation time τ, eigenfrequency ω_0 and damping coefficient (damping constant) q resp. r are related as follows:

$$Q = \omega_0 \cdot \tau = \frac{\omega_0}{2r}$$

$$q = \frac{2r}{\omega_0} = \frac{1}{Q}$$

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